J16/P13/4. Answers: $d = 6$, $r = 5$

The 1st, 3rd and 13th terms of an arithmetic progression are also the 1st, 2nd and 3rd terms respectively of a geometric progression. The first term of each progression is 3. Find the common difference of the arithmetic progression and the common ratio of the geometric progression. [5]

A.P

1st term $U_1 = a = 3$

3rd term $U_3 = 3 + 2d$

13th term $U_{13} = 3 + 12d$

G.P

$T_1 = U_1 = 3$

$T_2 = U_3 = 3 + 2d$

$T_3 = U_{12} = 3 + 12d$

$$r = \frac{T_2}{T_1} = \frac{3+2d}{3}$$

Also, $r = \frac{T_3}{T_2} = \frac{3+12d}{3+2d}$

$$\therefore \frac{3+2d}{3} = \frac{3+12d}{3+2d}$$

$$(3+2d)^2 = 3(3+12d)$$

$$9+12d+4d^2 = 9+36d$$

$$24d - 4d^2 = 0$$

$$4d(6-d) = 0$$

$$4d = 0 \quad or \quad 6-d = 0$$

$$d = 0 \qquad d = 6$$

$$d = 6 \text{ only}$$

$$r = \frac{3+2d}{3} = 5$$

J17/P11.4. Answers: (i) 50, (ii) 22400

(a) An arithmetic progression has a first term of 32, a 5th term of 22 and a last term of −28. Find the sum of all the terms in the progression. [4]

$$a = 32$$

$$a + 4d = 22$$

$$\therefore 32 + 4d = 22$$

$$4d = -10$$

$$d = -2.5$$

To use S_n formula, we need n also.
To find the number of terms,
we need to know which term is -28 (the last term).

Use T_n formula to do so.

A.P $T_n = -28$

$$a + (n-1)d = -28$$

$$32 - 2.5(n-1) = -28$$

$$2.5(n-1) = 60$$

$$(n-1) = 24$$

$$n = 25$$

$$\therefore S_n = \frac{25}{2}(32-28) = 50$$

(b) Each year a school allocates a sum of money for the library. The amount allocated each year increases by 2.5% of the amount allocated the previous year. In 2005 the school allocated $2000. Find the total amount allocated in the years 2005 to 2014 inclusive. [3]

Allocation 2005: $U_1 = 2000$

$$U_2 = 2000 + \left(\frac{2.5}{100} \times 2000\right)$$

Allocation 2006 : $= 2000(1+0.025)$

$$= 2000 \times 1.025$$

Allocation 2007:

$$U_3 = 2000 \times 1.025 + \left(\frac{2.5}{100} \times 2000 \times 1.025\right)$$

$$= 2000 \times 1.025(1+0.025)$$

$$= 2000 \times 1.025 \times 1.025$$

$$= 2000 \times 1.025^2$$

This is a G.P with $a = 2000$ and $r = 1.025$

2005 to 2014 inclusive = 2014 – 2005 +1 = 10 years

Total amount allocated = S_{10}

$$S_{10} = \frac{2000\left(1.025^{10}-1\right)}{1.025-1} = \$22400\,(3\text{ sf})$$

M17/P12/2. Answers: 8

In the expansion of $\left(\frac{1}{ax}+2ax^2\right)^5$, the coefficient of x is 5. Find the value of the constant a. [4]

Method 1:

$(r+1)^{\text{th}}$ term $= \binom{n}{r} a^{n-r} b^r$

in the expansion of $(a+b)^n$

$$\therefore (r+1)^{\text{th}} \text{ term} = \binom{5}{r}\left(\frac{1}{ax}\right)^{5-r}\left(2ax^2\right)^r$$

$$= \binom{5}{r}\frac{2^r a^r x^{2r}}{a^{5-r} x^{5-r}}$$

$$= \binom{5}{r} 2^r a^{2r-5} x^{3r-5}$$

Coefficient of x $\Rightarrow x^1$

$$\therefore x^{3r-5} = x^1$$

$$3r-5=1$$

$$r=2$$

Coefficient of x = $\binom{5}{2} 2^2 a^{-1} = 40a^{-1}$

$$\therefore 40a^{-1} = 5$$

$$a = 8$$

Method 2:

$$\left(\frac{1}{ax}+2ax^2\right)^5$$

$$= \left(\frac{1}{ax}\right)^5 + \binom{5}{1}\left(\frac{1}{ax}\right)^4\left(2ax^2\right)^1 + \binom{5}{2}\left(\frac{1}{ax}\right)^3\left(2ax^2\right)^2 \ldots$$

$$= \frac{1}{a^5x^5} + \frac{10}{a^3x^2} + \frac{40}{a}x \ldots$$

Coefficient of x = $\frac{40}{a}$

$$\therefore 40a^{-1} = 5$$

$$a = 8$$

M17/P12/8. Answers: (i) $2(3x+2)^2+3$ *, (ii)* $\frac{1}{3}\sqrt{(x-3)/2}-\frac{2}{3}$ *,* $x \geq 11$ *, (iii) 0, 4*

The functions f and g are defined for $x \geqslant 0$ by

$$f : x \mapsto 2x^2 + 3,$$
$$g : x \mapsto 3x + 2.$$

(i) Show that $gf(x) = 6x^2 + 11$ and obtain an unsimplified expression for $fg(x)$. [2]

$$gf(x) = g\left(2x^2+3\right)$$

$$= 3\left(2x^2+3\right)+2$$

$$= 6x^2+11$$

$$fg(x) = f(3x+2)$$

$$= 2(3x+2)^2+3$$

(ii) Find an expression for $(fg)^{-1}(x)$ and determine the domain of $(fg)^{-1}$. [5]

Let $y = fg(x)$

$$= 2(3x+2)^2+3$$

$$x = \frac{-2 \pm \sqrt{\frac{y-3}{2}}}{3}$$

The domain of fg is the domain of g: $x \geq 0$

Since x takes positive values only,

$$(\text{fg})^{-1}(x)=\frac{\sqrt{\frac{x-3}{2}}-2}{3}$$

Domain of $(\text{fg})^{-1}$ = Range of fg

$$\text{fg}(x)=2(3x+2)^2+3$$

Since domain of fg is $x\geq 0$, range of fg is $\text{fg}(x)\geq 11$

$\therefore$ Domain of $(\text{fg})^{-1}$ is $x\geq 11$

(iii) Solve the equation $\text{gf}(2x)=\text{fg}(x)$. [3]

$$\text{gf}(2x)=\text{fg}(x)$$
$$6(2x)^2+11=2(3x+2)^2+3$$
$$24x^2+11=18x^2+24x+11$$
$$6x^2-24x=0$$
$$6x(x-4)=0$$
$$x=0 \quad \text{or} \quad x=4$$

N16/P13/8. Answers: (i) $(2x+3)^2+1$*, (ii) 2x + 3 (iii)*$0.5\sqrt{x-1}-1.5$ *, x > 10*

(i) Express $4x^2+12x+10$ in the form $(ax+b)^2+c$, where a, b and c are constants. [3]

$$4x^2+12x+10$$
$$=4(x^2+3x+2.5)$$
$$=4\left[(x+1.5)^2+0.25\right]$$
$$=4(x+1.5)^2+1$$
$$=2^2(x+1.5)^2+1$$
$$=\left[2(x+1.5)\right]^2+1$$
$$=(2x+3)^2+1$$

(ii) Functions f and g are both defined for $x>0$. It is given that $\text{f}(x)=x^2+1$ and $\text{fg}(x)=4x^2+12x+10$. Find $\text{g}(x)$. [1]

$$\text{fg}(x)=4x^2+12x+10$$
$$\text{f}(\text{g}(x))=4x^2+12x+10$$
$$(\text{g}(x))^2+1=(2x+3)^2+1$$
$$\therefore \text{g}(x)=2x+3$$

(iii) Find $(\text{fg})^{-1}(x)$ and give the domain of $(\text{fg})^{-1}$. [4]

Let $y=\text{fg}(x)$

$$=4x^2+12x+10$$
$$=(2x+3)^2+1$$

$$x=\frac{-3\pm\sqrt{y-1}}{2}$$

fg has domain g: $x>0$,

$$\therefore (\text{fg})^{-1}(x)=\frac{-3+\sqrt{x-1}}{2}$$

Domain of $(\text{fg})^{-1}$ = Range of fg

Range of fg = $\text{fg}(x)>10$

$\therefore$ Domain of $(\text{fg})^{-1}$: $x>10$

N16/P12/3. Answers: (i) x < -1, x > 4, (ii) -3

A curve has equation $y=2x^2-6x+5$.

(i) Find the set of values of x for which $y>13$. [3]

$2x^2 - 6x + 5 > 13$

$2x^2 - 6x - 8 > 0$

$x^2 - 3x - 4 > 0$

$(x-4)(x+1) > 0$

Critical values of *x* = -1 , 4

	x < -1	-1 < x < 4	x > 4
$(x-4)(x+1)$	+	-	+

$\therefore x < -1 \cup x > 4$

(ii) Find the value of the constant k for which the line $y = 2x + k$ is a tangent to the curve. [3]

$y = 2x^2 - 6x + 5$

$y = 2x + k$

Replace *y* by *2x + k* into first equation

$2x + k = 2x^2 - 6x + 5$

$2x^2 - 8x + 5 - k = 0$

Since line is a tangent to the curve,

$b^2 - 4ac = 0$ for the above equation

$64 - 8(5-k) = 0$

$k = -3$

N16/P12/2. Answers: (i) k = 3, (ii) $-54.2°, 35.8°$

(i) Express the equation $\sin 2x + 3\cos 2x = 3(\sin 2x - \cos 2x)$ in the form $\tan 2x = k$, where k is a constant. [2]

$\sin 2x + 3\cos 2x = 3\sin 2x - 3\cos 2x$

$2\sin 2x = 6\cos 2x$

$\tan 2x = 3$

(ii) Hence solve the equation for $-90° \leqslant x \leqslant 90°$. [3]

$\tan 2x = 3$

$\text{Key angle} = \tan^{-1}(3) = 71.57° \quad (2 \text{ dp})$

$2x = 71.57°, 251.57°$

Now $-90° \le x \le 90°$, for 2*x* : $-180° \le 2x \le 180°$

To obtain additional answers $\pm 360°$ to $71.57°, 251.57°$

$\therefore 2x = 71.57°, -108.43°$ keeping only values of 2*x* for which $-180° \le 2x \le 180°$

$x = -54.2°, 35.8°$